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Compactification of the heterotic pure spinor superstring I

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Abstract: In this paper we begin the study of compactifications of the pure spinor formalism for superstrings. As a first example of such a process we study the case of the heterotic string in a Calabi-Yau background. We explicitly construct a BRST operator imposing $N = 1$ four-dimensional supersymmetry and show that nilpotence implies Kähler and Ricci-flatness conditions. The massless spectrum is computed using this BRST operator and it agrees with the expected result.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua

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Contents

1 Introduction

Since the realization that string theory could give rise to anomaly-free chiral theories, compactifications have been studied in many different contexts in attempts to make contact with the observed four-dimensional world. The process of compactification usually involves breaking of the extended supersymmetry present in higher dimensional supersymmetric theories.

The compactification procedure for the RNS superstring is well-known for backgrounds with pure NS fields. If one wants to include RR fields in the case of Type II string theories, then worldsheet methods are not available and one is forced to study it using only supergravity.

In the case of the RNS superstring, compactifications to Calabi-Yau manifolds and their orbifold limits are standard knowledge in the field and many interesting physical properties are derived using worldsheet methods. Alternative descriptions of the RNS superstring in compactified backgrounds, known as hybrid formalisms, were developed for two [1], four [2], and six [3] dimensions by Berkovits and collaborators. Since the roots of the hybrid formalism are in the RNS superstring, it was not known until recently [4] how to study compatifications with RR fields. One of the interesting aspects of the hybrid formalism approach to RR flux compactifications is that the $N = (2, 2)$ superconformal algebra, an essential ingredient in standard CY compactifications, is still preserved. On the other hand, a drawback of it is that it is not known how the procedure works if the starting point of the compactification is not a CY manifold. Furthermore, computations involving compactification-dependent states are subtle, and appropriate care should be taken in this case.[1]

For these reasons we would like to have another formalism in which it is possible to study more general flux compactifications. The pure spinor formalism [5] is the appropriate one. However, the pure spinor formalism has superspace coordinates corresponding to all supersymmetries and curved superspaces are not known explicitly except for maximally symmetric cases and the recent construction of the full Type IIA superspace for $AdS_4 \times \mathbb{C}P^3$ [6]. For the eleven-dimensional case a systematic procedure was developed in [7]. Although one could use this procedure, four-dimensional supersymmetry arguments are more effective to attack the present problem.

Compactifications of the pure spinor formalism is the theme of this paper. As a first step toward more general backgrounds in heterotic and type II theories, we will study compactifications of the heterotic string on a Calabi-Yau 3-fold. The pure spinor formalism was studied in cases with reduced supersymmetry previously in [8–12]. What is missed by some of these previous works is the input from the geometry of the Calabi-Yau and the full pure spinor constraint from ten dimensions. These two ingredients give extra terms to the BRST charge and these extra terms allow us to derive on-shell equations for the four-dimensional multiplets.

Chiral superspace and chiral coordinates. As an example of the construction in the next sections let us consider first the simple case of $N = 1$ four-dimensional supersymmetry. In this case the superspace coordinates are given by $(x^{\alpha\dot{\alpha}}, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})$. The supersymmetric derivatives are given by

$$D_\alpha = \partial_\alpha + i\overline{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \overline{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\alpha\partial_{\alpha\dot{\alpha}}. \tag{1.1}$$

It is well-known that a consistent non-trivial constraint on superfields is $\overline{D}_{\dot{\alpha}}\Phi = 0$. The easiest way to solve this constraint is to realize that the chiral variable $y^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + i\theta^\alpha\overline{\theta}^{\dot{\alpha}}$ is annihilated by $\overline{D}_{\dot{\alpha}}$, i.e. $\overline{D}_{\dot{\alpha}}y^{\beta\dot{\beta}} = 0$. We then construct superfields depending only on $(y^{\alpha\dot{\alpha}}, \theta^\alpha)$. Furthermore, the supersymmetric derivatives and supercharges in these variables are given by

$$\begin{aligned} D_\alpha &= \partial_\alpha + 2i\overline{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & \overline{D}_{\dot{\alpha}} &= \partial_{\dot{\alpha}}, \\ Q_\alpha &= \partial_\alpha, & \overline{Q}_{\dot{\alpha}} &= -\partial_{\dot{\alpha}} + 2i\theta^\alpha\partial_{\alpha\dot{\alpha}}, \end{aligned} \tag{1.2}$$

and we have that $Q_\alpha y^{\beta\dot{\beta}} = 0$. This means that any background field $\Phi(y^{\alpha\dot{\alpha}})$ is invariant under chiral supersymmetries.[2] Of course, in Minkowski signature, it is not possible to consider theories invariant only under the anti-chiral supersymmetry. (In a Euclidean signature the chiral and anti-chiral supersymmetries are not related by complex conjugation and such a symmetry is consistent.) The fact that the $y^{\alpha\dot{\alpha}}$ is not real forces us to include

[1] One of the authors (BCV) would like to thank Massimo Bianchi and Pierre Vanhove for pointing out these problems and for discussions on these issues.

[2] Note that we could also consider dependence on $\overline{\theta}$ but that is not a physical superfield, that is, not a representation of the supersymmetry algebra.

its complex conjugate and one should also consider functions which are not holomorphic in $y^{\alpha\dot{\alpha}}$. This means that chiral coordinates are not useful for reducing supersymmetry since we cannot have theories constructed only on subspace parameterized by $y^{\alpha\dot{\alpha}}$. (Of course the superpotential is a function on this subspace and one can use holomorphicity to prove non-renormalization theorems but there is also the D-term.)

It turns out that higher dimensional superspaces also have chiral-like variables. We will see that after we break ten-dimensional Lorentz invariance, type I supersymmetry in ten dimensions will have "chiral" variables invariant under four-dimensional $N = 1$ supersymmetry.

Organization. In the next section we introduce the pure spinor formalism and discuss general concepts that are useful in later sections. In section 3 we construct curved-space d-operators and the BRST charge for a complex six-dimensional internal manifold and show that nilpotence and four-dimensional supersymmetry require that the internal space is a Calabi-Yau manifold. Section 4 contains a discussion of the spectrum obtained from the cohomology of the BRST operator constructed in section 3. The final section contains future directions and open problems.

2 Preliminary concepts

In this section we discuss preliminary material needed for later sections. We begin with a short review of the pure spinor formalism. After that we discuss type I supersymmetry preserving only four dimensional Lorentz symmetry. We close this section with a review of complex geometry using frames.

2.1 Review of the pure spinor formalism

The action of the heterotic string in a flat background is given by

$$S = \int d^2 z \left[\frac{1}{2} \partial X^{\widehat{m}} \overline{\partial} X_{\widehat{m}} + p_{\widehat{\alpha}} \overline{\partial} \theta^{\widehat{\alpha}} + \omega_{\widehat{\alpha}} \overline{\partial} \lambda^{\widehat{\alpha}} \right] + S_\lambda + S_R, \tag{2.1}$$

where $(X^{\widehat{m}}, \theta^\alpha)$ parameterize the $D = 10$, $N = 1$ superspace and $p_{\widehat{\alpha}}$ is the fermionic conjugate momentum. S_λ is the action for the pure spinor $\lambda^{\widehat{\alpha}}$ which is defined to satisfy the constraint

$$\lambda \gamma^{\widehat{m}} \lambda = 0 \quad \text{for} \quad \widehat{m} = 0 \text{ to } 9. \tag{2.2}$$

Although an explicit form of S_λ in terms of λ and its conjugate momentum ω requires breaking SO$(9,1)$ (or its Euclidean version SO(10)) to a subgroup, the OPE of $\lambda^{\widehat{\alpha}}$ with its Lorentz current $N^{\widehat{m}\widehat{n}} = \frac{1}{2} \omega \gamma^{\widehat{m}\widehat{n}} \lambda$ is manifestly SO$(9,1)$ covariant. The condition (2.2) implies that ω is defined only up the gauge invariance

$$\delta \omega_{\widehat{\alpha}} = \Lambda^{\widehat{m}} (\gamma_{\widehat{m}} \lambda)_{\widehat{\alpha}}, \tag{2.3}$$

for any $\Lambda^{\widehat{m}}$. Finally, S_R is the action for the right-moving degrees of freedom which describe the reparametrization ghosts and the heterotic fermions.

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It is useful to define the supersymmetric operators in terms of the free worldsheet fields

$$d_{\widehat{\alpha}} = p_{\widehat{\alpha}} - \left(\Pi^{\widehat{m}} - \frac{1}{2}\theta\gamma^{\widehat{m}}\partial\theta\right)(\gamma_{\widehat{m}}\theta)_{\widehat{\alpha}}, \quad \Pi^{\widehat{m}} = \partial X^{\widehat{m}} + \theta\gamma^{\widehat{m}}\partial\theta, \tag{2.4}$$

which satisfy the OPE's

$$d_{\widehat{\alpha}}(y)d_{\widehat{\beta}}(z) \to -2\gamma^{\widehat{m}}_{\widehat{\alpha}\widehat{\beta}}\Pi_{\widehat{m}}(y-z)^{-1}, \quad d_{\widehat{\alpha}}(y)\Pi^{\widehat{m}}(z) \to (\gamma^{\widehat{m}}\partial\theta)_{\widehat{\alpha}}(y-z)^{-1}. \tag{2.5}$$

The BRST operator and left moving stress energy tensor are given by

$$Q = \oint \lambda^{\widehat{\alpha}} d_{\widehat{\alpha}}, \quad T = -\frac{1}{2}\partial X^{\widehat{m}}\partial X_{\widehat{m}} - p_{\widehat{\alpha}}\partial\theta^{\widehat{\alpha}} + T_\lambda \tag{2.6}$$

where $\lambda^{\widehat{\alpha}}$ carries ghost-number 1. Nilpotency is easily checked using the OPE's (2.5) and the pure spinor condition (2.2). It can be shown that the cohomological conditions give the equations of motion and gauge invariances of linearized $N=1, D=10$ supergravity.

In the right moving sector we have the heterotic fermions, $\overline{\Psi}_{\mathtt{A}}$, and the reparametrization ghosts, $(\overline{b}, \overline{c})$. The action for them is given by

$$S_R = \int d^2z \Big[\overline{\Psi}_{\mathtt{A}}\partial\overline{\Psi}_{\mathtt{A}} + \overline{b}\partial\overline{c}\Big]. \tag{2.7}$$

The right moving energy momentum tensor is

$$\overline{T} = -\frac{1}{2}\overline{\partial}X^{\widehat{m}}\overline{\partial}X_{\widehat{m}} - \overline{b}\overline{\partial}\overline{c} - \overline{\partial}(\overline{b}\overline{c}) + \overline{T}_{\mathbf{A}}, \tag{2.8}$$

where $\overline{T}_{\mathbf{A}}$ is the $c=16$ stress energy tensor coming from the heterotic fermions. Finally, the right moving BRST charge is given by

$$\overline{Q} = \oint \left(\overline{c}\,\overline{T} + \overline{c}\overline{\partial}\overline{c}\overline{b}\right). \tag{2.9}$$

Physical vertex operator should be in the cohomology of both Q and $\overline{Q}$.

The action in a general curved background can be constructed by adding the integrated vertex operator to the flat action of (2.1) and then covariantizing with respect to the background super-reparametrization invariance. The result of doing this is [13]

$$\begin{aligned} S = \int d^2z \frac{1}{2}\Pi^{\widehat{a}}\overline{\Pi}^{\widehat{b}}\eta_{\widehat{a}\widehat{b}} + \frac{1}{2}\Pi^{\widehat{A}}\overline{\Pi}^{\widehat{B}}B_{\widehat{B}\widehat{A}} + d_{\widehat{\alpha}}\overline{\Pi}^{\widehat{\alpha}} + \omega_{\widehat{\alpha}}\overline{\nabla}\lambda^{\widehat{\alpha}} + \overline{\Psi}_{\mathtt{A}}\nabla\overline{\Psi}_{\mathtt{A}} \\ + d_{\widehat{\alpha}}\overline{J}^I W_I^{\widehat{\alpha}} + \lambda^{\widehat{\alpha}}\omega_{\widehat{\beta}}\overline{J}^I U_{I\widehat{\alpha}}{}^{\widehat{\beta}} + S_{\text{FT}} + S_{bc}, \end{aligned} \tag{2.10}$$

where $\Pi^{\widehat{A}} = \partial Z^{\widehat{M}} E_{\widehat{M}}{}^{\widehat{A}}$ and $\overline{J}^I = \frac{1}{2}K^I_{\mathtt{AB}}\overline{\Psi}_{\mathtt{A}}\overline{\Psi}_{\mathtt{B}}$ with the Ks denoting the generators of the gauge group. The covariant derivatives are defined as

$$\overline{\nabla}\lambda^{\widehat{\alpha}} = \overline{\partial}\lambda^{\widehat{\alpha}} + \lambda^{\widehat{\beta}}\overline{\Omega}_{\widehat{\beta}}{}^{\widehat{\alpha}}, \quad \nabla\overline{\Psi}_{\mathtt{A}} = \partial\overline{\Psi}_{\mathtt{A}} + A_I K^I_{\mathtt{AB}}\overline{\Psi}_{\mathtt{B}},$$

where $\overline{\Omega}_{\widehat{\beta}}{}^{\widehat{\alpha}} = \overline{\Pi}^{\widehat{A}}\Omega_{\widehat{A}\widehat{\beta}}{}^{\widehat{\alpha}}, A_I = \Pi^{\widehat{A}}A_{I\widehat{A}}$ with $\Omega_{\widehat{A}\widehat{\beta}}{}^{\widehat{\alpha}}$ being the background connection for Lorentz and scaling transformations, and $A_{I\widehat{A}}$ the connection for background gauge transformations. The Fradkin-Tseytlin term S_{FT} is given by

$$S_{\text{FT}} = \frac{1}{2\pi}\int d^2z r\Phi, \tag{2.11}$$

where r is the world-sheet curvature and Φ is the dilaton superfield. Although this term is not necessary for having a covariant action, it is required to have a quantum conformally invariant sigma-model action [14, 15].

2.2 $N = 1$ ten dimensional supersymmetry

We are interested in a background preserving $N = 1$ supersymmetry in four dimensions. The corresponding supersymmetric derivative algebra is a sub-algebra of the ten dimensional supersymmetric derivative algebra. A 16-component, 10-dimensional spinor decomposes into

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\overline{\mathbf{2}}, \overline{\mathbf{4}}) \tag{2.12}$$

representations of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SU}(4)$. We will denote the four-dimensional coordinates as x^a or $x^{\alpha\dot{\alpha}}$ and the six dimensional coordinates by y^i where the index i goes from 1 to 6. To relate vector and spinor representations of the Lorentz group we use standard sigma matrices. The six dimensional sigma matrices are σ_i^{IJ} where $I, J = 1, \ldots, 4$ are SU(4) spinor indices and sigma is antisymmetric in I and J. These sigma matrices are related to the ones with indices down by

$$\overline{\sigma}^i_{IJ} = \frac{1}{2}\epsilon_{IJKL}\sigma^{i\,KL}. \tag{2.13}$$

Other useful identities that the six dimensional sigma matrices satisfy are

$$\sigma_i^{IJ}\overline{\sigma}^i_{KL} = \delta^I_K\delta^J_L - \delta^I_L\delta^J_K, \quad \sigma_i^{IJ}\sigma^{i\,KL} = \epsilon^{IJKL}. \tag{2.14}$$

The 16 supersymmetries are now parameterized by complex spinors $(\eta^I_\alpha, \overline{\eta}^{\dot{\alpha}}_I)$ and the worldsheet spinor variables are now $(\theta^I_\alpha, \overline{\theta}^{\dot{\alpha}}_I)$. The supersymmetry transformations of the bosonic variables are

$$\delta x^m = i\theta^I\sigma^m\overline{\eta}_I - i\eta^I\sigma^m\overline{\theta}_I \tag{2.15}$$

$$\delta y^i = i\theta^\alpha\overline{\sigma}^i\eta_\alpha - i\overline{\eta}_{\dot{\alpha}}\sigma^i\overline{\theta}^{\dot{\alpha}}, \tag{2.16}$$

where we suppressed the index contractions with the sigma matrices. As in four dimensions it is useful to consider

$$y^{IJ} = y^i\sigma_i^{IJ}, \quad \overline{y}_{IJ} = y_i\overline{\sigma}^i_{IJ}, \tag{2.17}$$

subject to the reality condition

$$(y^{IJ})^\dagger = \frac{1}{2}\epsilon^{IJKL}\overline{y}_{KL}, \tag{2.18}$$

inherited from (2.13).

Since we are interested in preserving only $N = 1$ supersymmetry in four dimensions, we split the SU(4) index to $(\mathsf{i}, \cdot)$ where $\mathsf{i} = 1$ to 3 and the $\cdot$ denotes a singlet under the SU(3) subgroup of SU(4). Now the odd superspace variables are $(\theta^\alpha, \overline{\theta}^{\dot{\alpha}}, \theta^{\mathsf{i}}_\alpha, \overline{\theta}^{\dot{\alpha}}_{\mathsf{i}})$ and the supersymmetry transformations are given by

$$\delta x^m = i\theta\sigma^m\eta - i\eta\sigma^m\theta + i\theta^{\mathsf{i}}\sigma^m\eta_{\mathsf{i}} - i\eta^{\mathsf{i}}\sigma^m\theta_{\mathsf{i}}, \tag{2.19}$$

$$\delta y^{\mathsf{i}} = i\theta^{\alpha\,\mathsf{i}}\eta_\alpha - i\theta^\alpha\eta^{\mathsf{i}}_\alpha - i\epsilon^{\mathsf{ijk}}\overline{\eta}_{\dot{\alpha}\mathsf{j}}\overline{\theta}^{\dot{\alpha}}_{\mathsf{k}}, \tag{2.20}$$

$$\delta y^{\mathsf{ij}} = i\theta^{\alpha\,\mathsf{i}}\eta^{\mathsf{j}}_\alpha - i\theta^{\alpha\,\mathsf{j}}\eta^{\mathsf{i}}_\alpha - i\epsilon^{\mathsf{ijk}}\overline{\eta}^{\dot{\alpha}}_{\mathsf{k}}\overline{\theta}_{\dot{\alpha}} + i\epsilon^{\mathsf{ijk}}\eta^{\dot{\alpha}}\theta_{\dot{\alpha}\,\mathsf{k}}. \tag{2.21}$$

Note that if we write y^{ij} as $\overline{y}^{\mathsf{i}} = \frac{1}{2}\epsilon^{\mathsf{ijk}} y^{\mathsf{jk}}$ the reality condition is just $(y^{\mathsf{i}})^\dagger = \overline{y}^{\mathsf{i}}$ which means that $(y^{\mathsf{i}}, \overline{y}^{\mathsf{i}})$ are usual complex coordinates. In the standard $\mathrm{SU}(4) \to \mathrm{SU}(3) \times U(1)$ decomposition, the spinors θ^{i} have U(1) charge $-\frac{1}{2}$ and the singlets θ have charge $\frac{3}{2}$ (and the opposite charges for the conjugate spinors). This is reflected in the supersymmetry transformations above since y^{i} has $+1$ charge.[3]

In this notation, the algebra of supersymmetric derivatives in flat space is given by

$$\begin{array}{lll} \{d_\alpha, d_\beta\} = 0 & \{d_\alpha, d_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha} & \{\overline{d}_{\dot\alpha}, \overline{d}_{\dot\beta}\} = 0 \\ \{d_\alpha, d_{\beta\mathsf{i}}\} = -2i\varepsilon_{\alpha\beta}\partial_{\mathsf{i}} & \{d_\alpha, \overline{d}_{\dot\alpha\mathsf{i}}\} = 0 & \{\overline{d}_{\dot\alpha}, \overline{d}_{\dot\beta\mathsf{i}}\} = -2i\varepsilon_{\dot\alpha\dot\beta}\overline{\partial}_{\mathsf{i}} \\ \{d_{\alpha\mathsf{i}}, d_{\beta\mathsf{j}}\} = -4i\varepsilon_{\alpha\beta}\epsilon_{\mathsf{ijk}}\overline{\partial}_{\mathsf{k}} & \{d_{\alpha\mathsf{i}}, \overline{d}_{\dot\alpha\mathsf{j}}\} = -2i\delta_{\mathsf{ij}}\partial_{\alpha\dot\alpha} & \{\overline{d}_{\dot\alpha\mathsf{i}}, \overline{d}_{\dot\beta\mathsf{j}}\} = -4i\varepsilon_{\dot\alpha\dot\beta}\epsilon_{\mathsf{ijk}}\partial_{\mathsf{k}}. \end{array} \tag{2.22}$$

A realization of this algebra in terms of the superspace coordinates is given by

$$\begin{aligned} d_\alpha &= \partial_\alpha + i\overline{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha} + i\theta^{\mathsf{i}}_\alpha\partial_{\mathsf{i}}, \\ \overline{d}_{\dot\alpha} &= -\overline{\partial}_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha} - i\overline{\theta}^{\mathsf{i}}_{\dot\alpha}\partial_{\mathsf{i}}, \\ d_{\alpha\mathsf{i}} &= \partial_{\alpha\mathsf{i}} + i\overline{\theta}^{\dot\alpha}_{\mathsf{i}}\partial_{\alpha\dot\alpha} - i\theta_\alpha\partial_{\mathsf{i}} - 2i\epsilon_{\mathsf{ijk}}\theta^{\mathsf{j}}_\alpha\overline{\partial}_{\mathsf{k}} \\ \overline{d}_{\dot\alpha\mathsf{i}} &= -\partial_{\dot\alpha\mathsf{i}} - i\theta^\alpha_{\mathsf{i}}\partial_{\alpha\dot\alpha} + i\overline{\theta}_{\dot\alpha}\overline{\partial}_{\mathsf{i}} + 2i\epsilon_{\mathsf{ijk}}\overline{\theta}^{\mathsf{j}}_{\dot\alpha}\partial_{\mathsf{k}} \end{aligned} \tag{2.23}$$

Since we are in flat space, there exist corresponding supercharges which commute with all these supersymmetric derivatives. However, as we will not need their full expression here, we will not write them.

The interesting property of the realization (2.23) using the notation described earlier is that there exist chiral-like coordinates analogous to the four-dimensional case described in the introduction:

$$z^{\mathsf{i}} = y^{\mathsf{i}} - i\theta^{\alpha\,\mathsf{i}}\theta_\alpha, \quad \overline{z}^{\mathsf{i}} = \overline{y}^{\mathsf{i}} - i\overline{\theta}^{\mathsf{i}}_{\dot\alpha}\overline{\theta}^{\dot\alpha}. \tag{2.24}$$

These are invariant under the SU(3) singlet supersymmetries generated by $(\eta^\alpha, \overline{\eta}^{\dot\alpha})$ but unlike the four-dimensional case, we can consistently consider functions of $(z^{\mathsf{i}}, \overline{z}^{\mathsf{i}})$ and still have Minkowski signature in spacetime. Furthermore, when written in these variables the realization (2.23) simplifies to

$$\begin{aligned} d_\alpha &= \partial_\alpha + i\overline{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha} + 2i\theta^{\mathsf{i}}_\alpha\partial_{\mathsf{i}}, \\ \overline{d}_{\dot\alpha} &= -\overline{\partial}_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha} - 2i\overline{\theta}^{\mathsf{i}}_{\dot\alpha}\partial_{\mathsf{i}}, \\ d_{\alpha\mathsf{i}} &= \partial_{\alpha\mathsf{i}} + i\overline{\theta}^{\dot\alpha}_{\mathsf{i}}\partial_{\alpha\dot\alpha} - 2i\epsilon_{\mathsf{ijk}}\theta^{\mathsf{j}}_\alpha\overline{\partial}_{\mathsf{k}} \\ \overline{d}_{\dot\alpha\mathsf{i}} &= -\partial_{\dot\alpha\mathsf{i}} - i\theta^\alpha_{\mathsf{i}}\partial_{\alpha\dot\alpha} + 2i\epsilon_{\mathsf{ijk}}\overline{\theta}^{\mathsf{j}}_{\dot\alpha}\partial_{\mathsf{k}}, \end{aligned} \tag{2.25}$$

where now the derivatives $(\partial_{\mathsf{i}}, \overline{\partial}_{\mathsf{i}})$ are taken with respect to $(z^{\mathsf{i}}, \overline{z}^{\mathsf{i}})$. Note that the algebra (2.22) is preserved. Furthermore, in these new variables, the corresponding supercharges for the supersymmetries generated by $(\eta^\alpha, \overline{\eta}^{\dot\alpha})$ are given by

$$q_\alpha = \partial_\alpha - \overline{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha}, \quad \overline{q}_{\dot\alpha} = -\partial_{\dot\alpha} + i\theta^\alpha\partial_{\alpha\dot\alpha}, \tag{2.26}$$

[3]Taking care to keep track of the U(1) charges, we can raise and lower all SU(3) indices at will with the understanding that we only apply Einstein summation convention when the index carriers have opposite U(1) charges.

which means that the variables $(z^{\mathsf{i}}, \overline{z}^{\mathsf{i}})$ are invariant under the SU(3) singlet supersymmetries. Note also that the new "chiral" variables $(z^{\mathsf{i}}, \overline{z}^{\mathsf{i}})$ are annihilated by

$$\overline{d}_{\dot\alpha} z^{\mathsf{i}} = 0, \quad d_{\alpha \mathsf{i}} z^{\mathsf{j}} = 0, \tag{2.27}$$

and this is consistent with the algebra (2.22). In other words, the constraints

$$\overline{d}_{\dot\alpha} \Psi = d_{\alpha \mathsf{i}} \Psi = 0 \tag{2.28}$$

on a general superfield Ψ are integrable.

In what follows, we will assume that our background fields depend only on these variables. Since the supercharges in (2.26) are independent of $(z^{\mathsf{i}}, \overline{z}^{\mathsf{i}})$ any background constructed with them will be invariant under this $N = 1$ supersymmetry. This also means that a background preserving this amount of supersymmetry is naturally almost-complex. Of course we still have to check that the background is on-shell. This will be the subject of section 3 where we will generalize the realization (2.25) to a curved six-dimensional background.

2.3 Complex and Kähler geometry using frames

The appropriate language to construct the pure spinor superstring sigma model in a general background uses frames. Since we want to study the heterotic string in a Calabi-Yau background, it is useful to review complex and Kähler geometry in this language. The reader familiar with this material, or willing to accept the interpretations of the relevant formulæ given in the subsequent sections, can skip ahead to section 3. This discussion is based on the definitions and conventions of [16].

A tangent complex index will be denoted by i, as in the previous subsection, and a coordinate (or "curved") index will be denoted by $\underline{\mathsf{i}}$. In a complex manifold of dimension n a hermitian metric is given in local coordinates by[4]

$$ds^2 = g_{\underline{\mathsf{i}}\underline{\mathsf{j}}} dz^{\underline{\mathsf{i}}} \otimes d\overline{z}^{\underline{\mathsf{j}}}. \tag{2.29}$$

The Riemannian metric on this manifold is given by $\mathrm{Re}(ds^2)$ and the imaginary part of ds^2 is given by

$$\omega = i g_{\underline{\mathsf{i}}\underline{\mathsf{j}}} dz^{\underline{\mathsf{i}}} \wedge d\overline{z}^{\underline{\mathsf{j}}}, \tag{2.30}$$

and is called the associated (1,1)-form (or Kähler form). An hermitian *coframe* is defined by two matrices $(E^{\mathsf{i}}{}_{\underline{\mathsf{i}}}, \overline{E}^{\mathsf{i}}{}_{\underline{\mathsf{j}}})$ such that

$$ds^2 = g_{\underline{\mathsf{i}}\underline{\mathsf{j}}} dz^{\underline{\mathsf{i}}} \otimes d\overline{z}^{\underline{\mathsf{j}}} = E^{\mathsf{i}}{}_{\underline{\mathsf{i}}} \overline{E}^{\mathsf{i}}{}_{\underline{\mathsf{j}}} dz^{\underline{\mathsf{i}}} \otimes d\overline{z}^{\underline{\mathsf{j}}} = E^{\mathsf{i}} \otimes \overline{E}^{\mathsf{i}}, \tag{2.31}$$

where $E^{\mathsf{i}} = E^{\mathsf{i}}{}_{\underline{\mathsf{i}}} dz^{\underline{\mathsf{i}}}$ and $\overline{E}^{\mathsf{i}} = \overline{E}^{\mathsf{i}}{}_{\underline{\mathsf{i}}} d\overline{z}^{\underline{\mathsf{i}}}$. Using the coframe, the associated (1,1)-form is given by

$$\omega = i E^{\mathsf{i}} \wedge \overline{E}^{\mathsf{i}}. \tag{2.32}$$

[4] A bar over an antiholomorphic index will not be used unless it is necessary.

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As usual, the exterior derivative is $d = \partial + \overline{\partial} = dz^{\underline{i}}\partial_{\underline{i}} + d\overline{z}^{\underline{i}}\overline{\partial}_{\underline{i}}$. We can compute the exterior derivative of the coframe giving

$$dE^{\mathrm{i}} = (dE^{\mathrm{i}}{}_{\underline{i}}) \wedge dz^{\underline{i}} = \left[(\overline{\partial}E^{\mathrm{i}}{}_{\underline{i}})E^{\underline{i}}{}_{\mathrm{j}} - \overline{E}^{\underline{i}}{}_{\mathrm{i}}\partial\overline{E}^{\mathrm{j}}{}_{\underline{i}}\right] \wedge E^{\mathrm{j}} + T^{\mathrm{i}}, \tag{2.33}$$

where T^{i} is a $(2,0)$-form defined by the equation above. Its explicit expression is

$$T^{\mathrm{i}} = (\partial E^{\mathrm{i}}{}_{\underline{i}})E^{\underline{i}}{}_{\mathrm{j}} \wedge E^{\mathrm{j}} + (\partial\overline{E}^{\mathrm{j}}{}_{\underline{i}})\overline{E}^{\underline{i}}{}_{\mathrm{i}} \wedge E^{\mathrm{j}} \tag{2.34}$$

with $E^{\underline{i}}{}_{\mathrm{i}} = (E^{\mathrm{i}}{}_{\underline{i}})^{-1}$. The complex manifold is *Kähler* if $T^{\mathrm{i}} = 0$. Equation (2.33) can be written in the form

$$dE^{\mathrm{i}} = \Omega^{\mathrm{i}}{}_{\mathrm{j}} \wedge E^{\mathrm{j}} + T^{\mathrm{i}} \tag{2.35}$$

where $\Omega^{\mathrm{i}}{}_{\mathrm{j}} = (\overline{\partial}E^{\mathrm{i}}{}_{\underline{i}})E^{\underline{i}}{}_{\mathrm{j}} - \overline{E}^{\underline{i}}{}_{\mathrm{i}}\partial\overline{E}^{\mathrm{j}}{}_{\underline{i}}$ and satisfies $\Omega + \overline{\Omega}^{\dagger} = 0$. Such a connection Ω is compatible with both the metric and complex structure. To see this more clearly, note that

$$d(E^{\mathrm{i}} \wedge \overline{E}^{\mathrm{i}}) = T^{\mathrm{i}} \wedge \overline{E}^{\mathrm{i}} - E^{\mathrm{i}} \wedge \overline{T}^{\mathrm{i}}. \tag{2.36}$$

From this last equation we can also see the standard definition of a Kähler manifold, that is, $d\omega = 0$ if $T^{\mathrm{i}} = 0$. The equations above allow us to define covariant exterior derivatives ∇ and $\overline{\nabla}$ given by

$$\nabla = \partial + (\overline{E}^{-1}\partial\overline{E})^{\mathrm{i}}{}_{\mathrm{j}}, \quad \overline{\nabla} = \overline{\partial} - (\overline{\partial}EE^{-1})^{\mathrm{i}}{}_{\mathrm{j}}. \tag{2.37}$$

With this definition we can say that E^{i} is covariantly holomorphic

$$\overline{\nabla}E^{\mathrm{i}} = 0, \tag{2.38}$$

while the holomorphic covariant exterior derivative ∇ defines the torsion

$$\nabla E^{\mathrm{i}} = T^{\mathrm{i}}, \tag{2.39}$$

In the case of vanishing torsion, these last two equations say

$$\overline{\nabla}_{\underline{i}}E^{\mathrm{i}}{}_{\underline{j}} = 0, \qquad \nabla_{\underline{i}}E^{\mathrm{i}}{}_{\underline{j}} = \nabla_{\underline{j}}E^{\mathrm{i}}{}_{\underline{i}}, \tag{2.40}$$

where the second equation translates to the usual $\partial_{\underline{i}}g_{\underline{j}\underline{k}} = \partial_{\underline{j}}g_{\underline{i}\underline{k}}$. One should be careful to note that the definition of covariant derivatives acts differently on the frames E^{i} and $\overline{E}^{\mathrm{i}}$, i.e. $(\nabla)^{\dagger} \neq \overline{\nabla}$. This is because the connection Ω defined above is skew-hermitian:

$$\Omega + \Omega^{\dagger} = 0 \rightarrow (d + \Omega)^{\dagger} = d - \Omega, \tag{2.41}$$

so the analogous expressions for the covariant derivatives (2.37) for $\overline{E}^{\mathrm{i}}$ have opposite signs and we have, in the case of vanishing $\overline{T}^{\mathrm{i}}$,

$$\nabla_{\underline{i}}\overline{E}^{\mathrm{i}}{}_{\underline{j}} = 0, \qquad \overline{\nabla}_{\underline{i}}\overline{E}^{\mathrm{i}}{}_{\underline{j}} = \overline{\nabla}_{\underline{j}}\overline{E}^{\mathrm{i}}{}_{\underline{i}}. \tag{2.42}$$

Curvature. We can define new covariant derivatives using the inverse of the coframe matrices.

$$\nabla_{i} = E^{\underline{i}}{}_{i}\nabla_{\underline{i}}, \quad \overline{\nabla}_{i} = \overline{E}^{\underline{i}}{}_{i}\overline{\nabla}_{\underline{i}}. \tag{2.43}$$

Also, note that because of (2.40) we have

$$\overline{\nabla}_{\underline{i}}E^{\underline{j}}{}_{i} = 0, \qquad E^{i}{}_{\underline{j}}\nabla_{\underline{i}}E^{\underline{k}}{}_{i} = E^{i}{}_{\underline{i}}\nabla_{\underline{j}}E^{\underline{k}}{}_{i}. \tag{2.44}$$

Using the new covariant derivatives above we can rewrite these expressions as

$$\overline{\nabla}_{j}E^{\underline{j}}{}_{i} = 0, \qquad \nabla_{i}E^{\underline{k}}{}_{j} = \nabla_{j}E^{\underline{k}}{}_{i}. \tag{2.45}$$

As usual, the curvature is defined from the commutators of covariant derivatives. Since the manifold is hermitian, we have $[\nabla_{\underline{i}}, \nabla_{\underline{j}}] = 0$. The same will be true for ∇_{i} precisely if the second equation in (2.45) holds. So, in terms of the new covariant derivatives the Kähler condition is $[\nabla_{i}, \nabla_{j}] = 0$. We will see how this condition arises from nilpotence of the BRST charge in section 3.

The non vanishing part of the curvature matrix can be defined as

$$R_{\underline{i}\bar{\underline{j}}} = [\nabla_{\underline{i}}, \overline{\nabla}_{\underline{j}}]. \tag{2.46}$$

Since the first equation of (2.45) holds we have that

$$R_{i\bar{j}} = E^{\underline{i}}{}_{i}\overline{E}^{\underline{j}}{}_{j}R_{\underline{i}\bar{\underline{j}}} = [\nabla_{i}, \overline{\nabla}_{j}]. \tag{2.47}$$

Due to all the symmetries the curvature matrix has when the manifold is Kähler, there are three equivalent ways to write the Ricci-flatness condition. The first is the usual $\mathrm{Ric} = 0$, the second is $\mathrm{Tr}(R_{i\bar{j}}) = 0$ and the last one is $\delta^{ij}R_{i\bar{j}} = \delta^{ij}[\nabla_{i}, \overline{\nabla}_{j}] = 0$. Again, in section 3 we will see how this last equation appears from nilpotence of the BRST charge.

Vector bundles. Since we are studying the heterotic string, we know vector bundles also appear in the theory and couple to the background in a non-trivial way. Consider that our manifold comes with additional structure given by a gauge 1-form

$$\mathcal{A} = \mathtt{A}_{i}E^{i} + \overline{\mathtt{A}}_{i}\overline{E}^{i} = A^{\Sigma}_{i}\mathbf{T}_{\Sigma}E^{i} + \overline{A}^{\Sigma}_{i}\mathbf{T}_{\Sigma}\overline{E}^{i}, \tag{2.48}$$

where $\mathbf{T}_{\Sigma}$ are the gauge algebra generators. We generalize the covariant derivatives above to include this gauge 1-form connection

$$\mathcal{D}_{i} = \nabla_{i} - \mathtt{A}_{i}, \qquad \overline{\mathcal{D}}_{i} = \overline{\nabla}_{i} - \overline{\mathtt{A}}_{i}. \tag{2.49}$$

Computing again the conditions that give Kähler and Ricci-flatness $[\mathcal{D}_{i}, \mathcal{D}_{j}] = 0$ and $\delta^{ij}[\mathcal{D}_{i}, \overline{\mathcal{D}}_{j}] = 0$ they factorize into original Kähler and Ricci-flatness and holomorphic YM equations, i.e. $F_{ij} = 0$ and $\delta^{ij}F_{i\bar{j}} = 0$.

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3 An Ansatz for the d-operator algebra and BRST charge

The expression for the d-operators in a general curved background was derived in [13]. It is given by

$$d_{\widehat{\alpha}} = E_{\widehat{\alpha}}{}^{\widehat{M}}\Big[P_{\widehat{M}} + \frac{1}{2}B_{\widehat{M}\widehat{N}}(\partial Z^{\widehat{N}} - \overline{\partial} Z^{\widehat{n}}) - \Omega_{\widehat{M}}{}^{\widehat{\beta}}{}_{\widehat{\gamma}}\lambda^{\widehat{\gamma}}\omega_{\widehat{\beta}} - \mathtt{A}_{\widehat{M}}^{\Sigma}\mathbf{J}_{\Sigma}\Big], \tag{3.1}$$

where $P_{\widehat{M}}$ are the momenta conjugate to the worldsheet variables defined as $P_{\widehat{M}} = \delta S/\delta(\partial_0 Z^{\widehat{M}})$. The nilpotence of the BRST charge is computed using Poison brackets $[P_{\widehat{M}}, Z^{\widehat{N}}]_{rmPB} = \delta_{\widehat{M}}^{\widehat{N}}$ and $[\lambda^{\widehat{\alpha}}, \omega_{\widehat{\beta}}]_{\mathrm{PB}} = \delta_{\widehat{\beta}}^{\widehat{\alpha}}$. Note that the background field $B_{\widehat{M}\widehat{N}}$ does not mix with the other background fields in (3.1) when we compute the nilpotence condition. This mixing only occurs when computing holomorphicity of the BRST current. In a flat background the d operator reduces to $d_{\widehat{\alpha}} = E_{\widehat{\alpha}}{}^{\widehat{M}}P_{\widehat{M}}$ (ignoring the contribution from the flat $B_{\widehat{M}\widehat{N}}$) and using the expression for the flat frame in the $4+6$ notation we get precisely (2.23) after replacing the conjugate momenta by the corresponding derivatives. The flat space BRST charge is

$$Q = \oint \Big(\lambda^{\alpha}d_{\alpha} + \overline{\lambda}^{\dot{\alpha}}\overline{d}_{\dot{\alpha}} + \lambda^{\alpha\,\mathsf{i}}d_{\alpha\,\mathsf{i}} + \overline{\lambda}^{\dot{\alpha}\,\mathsf{i}}\overline{d}_{\dot{\alpha}\,\mathsf{i}}\Big), \tag{3.2}$$

and it will square to zero if the ghosts satisfy the pure spinor constraint, reduced to $4+6$ notation

$$\lambda^{\alpha}\overline{\lambda}^{\dot{\alpha}} + \lambda^{\alpha\,\mathsf{i}}\overline{\lambda}^{\dot{\alpha}\,\mathsf{i}} = 0, \tag{3.3}$$
$$\lambda^{\alpha}\lambda_{\alpha}^{\mathsf{i}} - \epsilon^{\mathsf{ijk}}\overline{\lambda}_{\dot{\alpha}}^{\mathsf{j}}\overline{\lambda}^{\dot{\alpha}\,\mathsf{k}} = 0, \tag{3.4}$$
$$\overline{\lambda}^{\dot{\alpha}}\overline{\lambda}_{\dot{\alpha}}^{\mathsf{i}} - \epsilon^{\mathsf{ijk}}\lambda_{\alpha}^{\mathsf{j}}\overline{\lambda}^{\alpha\,\mathsf{k}} = 0. \tag{3.5}$$

We want to generalize this to a flat four-dimensional background plus a curved six dimensional one. We must find the appropriate generalization of the d operators for this case. The first thing to note is that if they are generalized to covariant derivatives $(\nabla_{\alpha}, \overline{\nabla}_{\dot{\alpha}}, \nabla_{\alpha\,\mathsf{i}}, \overline{\nabla}_{\dot{\alpha}\,\mathsf{i}})$ satisfying the following algebra

$$\begin{array}{lll}
\{\nabla_{\alpha}, \nabla_{\beta}\} = 0 & \{\nabla_{\alpha}, \overline{\nabla}_{\dot{\alpha}}\} = -2i\nabla_{\alpha\dot{\alpha}} & \{\overline{\nabla}_{\dot{\alpha}}, \overline{\nabla}_{\dot{\beta}}\} = 0 \\
\{\nabla_{\alpha}, \nabla_{\beta\,\mathsf{i}}\} = -2i\varepsilon_{\alpha\beta}\nabla_{\mathsf{i}} & \{\nabla_{\alpha}, \overline{\nabla}_{\dot{\alpha}\,\mathsf{i}}\} = 0 & \{\overline{\nabla}_{\dot{\alpha}}, \overline{\nabla}_{\dot{\beta}\,\mathsf{i}}\} = -2i\varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\nabla}_{\mathsf{i}} \\
\{\nabla_{\alpha\,\mathsf{i}}, \nabla_{\beta\,\mathsf{j}}\} = -4i\varepsilon_{\alpha\beta}\epsilon_{\mathsf{ijk}}\overline{\nabla}_{\mathsf{k}} & \{\nabla_{\alpha\,\mathsf{i}}, \overline{\nabla}_{\dot{\alpha}\,\mathsf{j}}\} = -2i\delta_{\mathsf{i}\overline{\mathsf{j}}}\nabla_{\alpha\dot{\alpha}} & \{\overline{\nabla}_{\dot{\alpha}\,\mathsf{i}}, \overline{\nabla}_{\dot{\beta}\,\mathsf{j}}\} = -4i\varepsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\overline{\mathsf{ijk}}}\nabla_{\mathsf{k}}
\end{array} \tag{3.6}$$

the BRST charge will be nilpotent. Here, the covariant derivatives $(\nabla_{\alpha\dot{\alpha}}, \nabla_{\mathsf{i}}, \overline{\nabla}_{\mathsf{i}})$ are defined by these equations. Using the variables defined in section 2.2 we can write the spinor covariant derivatives as

$$\begin{aligned}
\nabla_{\alpha} &= \partial_{\alpha} + i\overline{\theta}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + 2i\theta_{\alpha}^{\mathsf{i}}\nabla_{\mathsf{i}}, \\
\overline{\nabla}_{\dot{\alpha}} &= -\overline{\partial}_{\dot{\alpha}} - i\theta^{\alpha}\nabla_{\alpha\dot{\alpha}} - 2i\overline{\theta}_{\dot{\alpha}}^{\mathsf{i}}\nabla_{\mathsf{i}}, \\
\nabla_{\alpha\,\mathsf{i}} &= \partial_{\alpha\,\mathsf{i}} + i\overline{\theta}_{\mathsf{i}}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} - 2i\epsilon_{\mathsf{ijk}}\theta_{\alpha}^{\mathsf{j}}\overline{\nabla}_{\mathsf{k}} \\
\overline{\nabla}_{\dot{\alpha}\,\mathsf{i}} &= -\partial_{\dot{\alpha}\,\mathsf{i}} - i\theta_{\mathsf{i}}^{\alpha}\nabla_{\alpha\dot{\alpha}} + 2i\epsilon_{\mathsf{ijk}}\overline{\theta}_{\dot{\alpha}}^{\mathsf{j}}\nabla_{\mathsf{k}},
\end{aligned} \tag{3.7}$$

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The higher order dependence on θs come from the derivatives $(\nabla_{\alpha\dot{\alpha}}, \nabla_{\mathsf{i}}, \overline{\nabla}_{\mathsf{i}})$. Note that the equations (3.7) can be put in the form (3.1) with the spin connection term $\Omega_{\widehat{M}}{}^{\widehat{\beta}}{}_{\widehat{\gamma}}\lambda^{\widehat{\gamma}}\omega_{\widehat{\beta}}$ and the gauge connection term $\mathtt{A}_{\widehat{M}}^{\Sigma}\mathbf{J}_{\Sigma}$ inside the bosonic covariant derivatives. Since the background does not break four-dimensional Lorentz symmetry, the covariant derivative $\nabla_{\alpha\dot{\alpha}}$ is just $\partial_{\alpha\dot{\alpha}} + \mathcal{O}(\theta^2)$ and nothing will depend on $x^{\alpha\dot{\alpha}}$. Moreover, since we are imposing that the background is invariant under the $N = 1$ supersymmetry, the background cannot depend on $(\theta^{\alpha}, \overline{\theta}^{\dot{\alpha}})$. We will now derive the restrictions imposed by these conditions.

Repeated application of the Jacobi identities

$$(-)^{AC}[\nabla_A, [\nabla_B, \nabla_C\}\} + (-)^{BA}[\nabla_B, [\nabla_C, \nabla_A\}\} + (-)^{CB}[\nabla_C, [\nabla_A, \nabla_B\}\} = 0,$$

for the covariant derivatives will show that the background is on-shell. Here A, B and C corresponds to any tangent space index. At dimension 3/2 we have

$$[\nabla_{\alpha}, \nabla_{\beta\dot{\beta}}] = \varepsilon_{\alpha\beta}\overline{W}_{\dot{\beta}}, \qquad [\nabla_{\alpha}, \overline{\nabla}_{\mathsf{i}}] = \overline{F}_{\alpha\mathsf{i}}, \qquad [\overline{\nabla}_{\dot{\alpha}}, \overline{\nabla}_{\mathsf{i}}] = 0, \tag{3.8}$$

together with their complex conjugates. Note that the first and last equations are a consequence of the algebra (3.6) plus Jacobi identities, while the second is the definition of $\overline{F}_{\alpha\mathsf{i}}$. To proceed, we have to solve order-by-order in θs using the Jacobi identities. Four-dimensional Lorentz invariance implies that the first components of the superfields defined above vanish and their second components should be four-dimensional scalars, as discussed above. The field-strengths $(W_{\alpha}, \overline{F}_{\alpha\mathsf{i}})$ have an expansion in powers of θs. In particular we have the components

$$W_{\alpha} = \theta_{\alpha}\mathrm{D} + \theta^{\mathsf{i}}_{\alpha}h_{\mathsf{i}} + \ldots \qquad \overline{F}_{\alpha\mathsf{i}} = \theta_{\alpha}\overline{\mathrm{F}}_{\mathsf{i}} + \theta^{\mathsf{j}}_{\alpha}R_{\bar{\mathsf{i}}\mathsf{j}} + \ldots \tag{3.9}$$

where the ellipses denote components that do not concern us at the moment. The background defined by (3.6) will be $N = 1$ supersymmetric if and only if these components vanish since all field-strengths should be invariant under shifts of $(\theta^{\alpha}, \overline{\theta}^{\dot{\alpha}})$. This is related to the usual $N = 1$ field theory requirement that in order to have a supersymmetric vacuum, D and F terms should vanish. The h_{i} and $R_{\bar{\mathsf{i}}\mathsf{j}}$ components are, at this stage, not required to vanish and are related to the geometry of the compactified space. We will now calculate the values of these components in terms of higher-dimension field-strengths.

Using the Jacobi identities again we can alternative forms of the field-strengths:

$$[\nabla_{\alpha\mathsf{i}}, \nabla_{\beta\dot{\beta}}] = \varepsilon_{\alpha\beta}F_{\dot{\beta}\mathsf{i}}, \qquad [\nabla_{\alpha\mathsf{i}}, \nabla_{\mathsf{j}}] = -2\epsilon_{\mathsf{ijk}}\overline{F}_{\alpha\mathsf{k}}, \qquad [\nabla_{\alpha\mathsf{i}}, \overline{\nabla}_{\mathsf{j}}] = -\delta_{\mathsf{i}\bar{\mathsf{j}}}W_{\alpha}. \tag{3.10}$$

At lowest order in θ the $\overline{\mathrm{F}}_{\mathsf{i}}$ component inside $\overline{F}_{\alpha\mathsf{i}}$ is given by $\{\nabla_{\alpha}, \overline{F}_{\beta\mathsf{i}}\} = \varepsilon_{\alpha\beta}\overline{\mathrm{F}}_{\mathsf{i}}$. However, using (3.10) we can write $\overline{F}_{\alpha\mathsf{i}}$ as

$$\overline{F}_{\alpha\mathsf{i}} = \frac{1}{2}\epsilon_{\mathsf{ijk}}[\nabla_{\alpha\mathsf{j}}, \nabla_{\mathsf{k}}]. \tag{3.11}$$

Now, the $\{\nabla_{\alpha}, [\nabla_{\beta\mathsf{j}}, \nabla_{\mathsf{k}}]\}$ Jacobi identity implies that

$$\overline{\mathrm{F}}_{\mathsf{i}} = \frac{i}{2}\epsilon_{\mathsf{ijk}}[\nabla_{\mathsf{j}}, \nabla_{\mathsf{k}}] \tag{3.12}$$

and since $[\nabla_{\mathsf{j}}, \nabla_{\mathsf{k}}]$ is anti-symmetric, it follows that the vanishing of the component $\overline{\mathrm{F}}_{\mathsf{i}}$ implies that $[\nabla_{\mathsf{j}}, \nabla_{\mathsf{k}}] = 0$. As we saw in section 2, these two conditions imply that the compactification manifold is Kähler and that the vector bundle over it is holomorphic.

In a similar way, the component D of W_α is the lowest component of $\{\nabla_\alpha, W_\beta\} = \varepsilon_{\alpha\beta}\mathrm{D}$. The computation of its value in terms of higher dimension field-strengths has one additional step. First we have to use the Jacobi identity with $\{\nabla_\alpha, [\nabla_{\beta\mathsf{i}}, \overline{\nabla}_{\mathsf{j}}]\}$ to find

$$\delta_{\mathsf{i}\bar{\mathsf{j}}}\{\nabla_\alpha, W^\alpha\} = \{\nabla_{\alpha\mathsf{i}}, \overline{F}^{\alpha}_{\mathsf{j}}\} + 4i[\nabla_{\mathsf{i}}, \overline{\nabla}_{\mathsf{j}}] \tag{3.13}$$

Next, we use the Jacobi identity with $\{\nabla_{\alpha\mathsf{i}}, [\nabla_{\beta_{\mathsf{j}}}, \nabla_{\mathsf{k}}]\}$ to find

$$\{\nabla_{\alpha\mathsf{i}}, \overline{F}^{\alpha}_{\mathsf{j}}\} = -4i[\nabla_{\mathsf{i}}, \overline{\nabla}_{\mathsf{j}}] + 2i\delta_{\mathsf{i}\bar{\mathsf{j}}}\delta^{\mathsf{k}\bar{\mathsf{l}}}[\nabla_{\mathsf{k}}, \overline{\nabla}_{\mathsf{l}}]. \tag{3.14}$$

Plugging this result back into (3.13) we find

$$\{\nabla_\alpha, W^\alpha\} = 2i\delta^{\mathsf{k}\bar{\mathsf{l}}}[\nabla_{\mathsf{k}}, \overline{\nabla}_{\mathsf{l}}]. \tag{3.15}$$

This means that the D component of W_α vanishes when $\delta^{\mathsf{k}\bar{\mathsf{l}}}[\nabla_{\mathsf{k}}, \overline{\nabla}_{\mathsf{l}}] = 0$. This equation is the second condition imposed by four-dimensional supersymmetry.

In summary, we have found that the vanishing of F-terms in the superfield $\overline{F}_{\alpha\mathsf{i}}$ implies the Kähler condition on the compactified manifold and part of holomorphic YM equations for the gauge background. The vanishing of the D-term in the W_α field-strength implies Ricci-flatness and the remaining equation for the set of holomorphic YM equations. One can proceed to find the values of the other components of the field-strengths and compute the expression for the curved d-operators in (3.6) explicitly. For example, one can use the Jacobi identity with $\{\nabla_{\alpha\mathsf{i}}, [\nabla_{\beta\mathsf{j}}, \nabla_{\mathsf{k}}]\}$ to find that $R_{\bar{\mathsf{i}}\mathsf{j}} = -2i[\nabla_{\mathsf{j}}, \overline{\nabla}_{\mathsf{i}}]$. The component h_{i} of W_α vanishes due to the Kähler condition and the Jacobi identity with $\{\nabla_{\alpha\mathsf{i}}, [\nabla_{\beta\mathsf{j}}, \overline{\nabla}_{\mathsf{k}}]\}$.

4 Physical state conditions and spectrum

Now that we have a BRST operator for the compactified background we want to check that $Q(V) = 0$ on a ghost number one vertex operator V gives the correct spectrum for the compactification. In order to do this we will first show that we get the correct equations of motion for a super-Maxwell multiplet plus three chiral fields with $N = 1$ supersymmetry in four dimensions and then generalize to the full string.

4.1 Ten dimensional super-YM in $1+3$ notation

The propose of this section is to see how standard $N = 1$ superfield equations of motion appear when we perform a toroidal reduction of the ten dimensional ghost number one vertex operator and BRST charge. The vertex operator takes the form

$$V = \lambda^\alpha A_\alpha + \overline{\lambda}^{\dot\alpha}\overline{A}_{\dot\alpha} + \lambda^{\alpha\mathsf{i}}A_{\alpha\mathsf{i}} + \overline{\lambda}^{\dot\alpha\mathsf{i}}\overline{A}_{\dot\alpha\mathsf{i}} \tag{4.1}$$

where $(A_\alpha, \overline{A}_{\dot\alpha}, A_{\alpha\mathsf{i}}, \overline{A}_{\dot\alpha\mathsf{i}})$ are superfields of the full superspace. The solution of $QV = 0$ where Q is given by equation (3.2) is

$$\begin{aligned}
d_\alpha A_\beta + d_\beta A_\alpha &= 0 \\
d_\alpha A_{\beta\mathsf{i}} + d_{\beta\mathsf{i}} A_\alpha &= \varepsilon_{\alpha\beta}\overline{\Phi}_{\mathsf{i}} \\
d_{\alpha\mathsf{i}} A_{\beta\mathsf{j}} + d_{\beta\mathsf{j}} A_{\alpha\mathsf{i}} &= 2\varepsilon_{\alpha\beta}\epsilon_{\mathsf{ijk}}\Phi_{\mathsf{k}} \\
d_\alpha \overline{A}_{\dot\alpha} + \overline{d}_{\dot\alpha} A_\alpha &= A_{\alpha\dot\alpha} \\
d_{\alpha\mathsf{i}} \overline{A}_{\dot\alpha\mathsf{j}} + \overline{d}_{\dot\alpha\mathsf{j}} A_{\alpha\mathsf{i}} &= \delta_{\mathsf{i}\bar{\mathsf{j}}} A_{\alpha\dot\alpha} \\
\overline{d}_{\dot\alpha} \overline{A}_{\dot\beta} + \overline{d}_{\dot\beta} \overline{A}_{\dot\alpha} &= 0 \\
\overline{d}_{\dot\alpha} \overline{A}_{\dot\beta\mathsf{i}} + \overline{d}_{\dot\beta\mathsf{i}} \overline{A}_{\dot\alpha} &= \varepsilon_{\dot\alpha\dot\beta}\Phi_{\mathsf{i}} \\
\overline{d}_{\dot\alpha\mathsf{i}} \overline{A}_{\dot\beta\mathsf{j}} + \overline{d}_{\dot\beta\mathsf{j}} \overline{A}_{\dot\alpha\mathsf{i}} &= 2\varepsilon_{\dot\alpha\dot\beta}\epsilon_{\mathsf{ijk}}\overline{\Phi}_{\mathsf{k}} \\
d_\alpha \overline{A}_{\dot\beta\mathsf{i}} + \overline{d}_{\dot\beta\mathsf{i}} A_\alpha &= 0 \\
\overline{d}_{\dot\alpha} A_{\beta\mathsf{i}} + d_{\beta\mathsf{i}} \overline{A}_{\dot\alpha} &= 0,
\end{aligned} \tag{4.2}$$

where the ds are defined in (2.25) and $(A_{\alpha\dot\alpha}, \Phi_{\mathsf{i}}, \overline{\Phi}_{\mathsf{i}})$ are defined by these equations. The vertex operator V also has the gauge invariance $\delta V = Q\Lambda$ with a real superfield Λ. In terms of its components, this translates to

$$\delta A_\alpha = d_\alpha \Lambda, \quad \delta A_{\alpha\mathsf{i}} = d_{\alpha\mathsf{i}}\Lambda \tag{4.3}$$

together with their complex conjugates. The first equation in (4.2) implies that $A_\alpha = d_\alpha V$ for some complex superfield V. The equations of motion imply the following gauge invariance

$$\delta A_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha}\Lambda, \quad \delta\Phi_{\mathsf{i}} = \overline{\partial}_{\mathsf{i}}\Lambda, \quad \delta\overline{\Phi}_{\mathsf{i}} = \partial_{\mathsf{i}}\Lambda \tag{4.4}$$

We can use the algebra of the supersymmetric derivatives to derive various relations on the fields defined by (4.2). It is possible to solve all the Bianchi identities for a general set of $(A_{\alpha\dot\alpha}, \Phi_{\mathsf{i}}, \overline{\Phi}_{\mathsf{i}})$ but since our goal is to generalize this to the case of a CY compactification, we will take another route. First, note that it is possible to fix $(A_\alpha, \overline{A}_{\dot\alpha})$ to vanish without trivializing the system of equations. The gauge transformation that preserves this choice has to satisfy

$$d_\alpha \Lambda = \overline{d}_{\dot\alpha}\Lambda = 0, \tag{4.5}$$

which, by use of the d-operator algebra, means Λ is just a constant in four dimensions. This implies that the degrees of freedom described by $(0, 0, A_{\alpha\mathsf{i}}, \overline{A}_{\dot\alpha\mathsf{j}})$ do not have gauge invariance from the four-dimensional point of view.

When $A_\alpha = \overline{A}_{\dot\alpha} = 0$ the equations (4.2) simplify to

$$\begin{aligned}
d_\alpha A_{\beta\mathsf{i}} &= \varepsilon_{\alpha\beta}\overline{\Phi}_{\mathsf{i}}, & \overline{d}_{\dot\alpha} A_{\beta\mathsf{i}} &= 0 \\
d_{\alpha\mathsf{i}} A_{\beta\mathsf{j}} + d_{\beta\mathsf{j}} A_{\alpha\mathsf{i}} &= 2\varepsilon_{\alpha\beta}\epsilon_{\mathsf{ijk}}\Phi_{\mathsf{k}}, & d_{\alpha\mathsf{i}} \overline{A}_{\dot\alpha\mathsf{j}} + \overline{d}_{\dot\alpha\mathsf{j}} A_{\alpha\mathsf{i}} &= 0 \\
\overline{d}_{\dot\alpha\mathsf{i}} \overline{A}_{\dot\beta\mathsf{j}} + \overline{d}_{\dot\beta\mathsf{j}} \overline{A}_{\dot\alpha\mathsf{i}} &= 2\varepsilon_{\dot\alpha\dot\beta}\epsilon_{\mathsf{ijk}}\overline{\Phi}_{\mathsf{k}} & \overline{d}_{\dot\alpha} \overline{A}_{\dot\beta\mathsf{i}} &= \varepsilon_{\dot\alpha\dot\beta}\Phi_{\mathsf{i}}
\end{aligned} \tag{4.6}$$

The first equation can be used to show that $d_\alpha \overline{\Phi}_{\mathsf{i}} = 0$, so it describes an anti-chiral field. The second equation together with the first shows that $\overline{d}^2\overline{\Phi}_{\mathsf{i}} = 0$, which is the

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massless equation for a chiral field. Then, using the commutator $[\overline{d}^2, \overline{d}_{\dot\alpha\,\mathsf{i}}] = -4i\overline{d}_{\dot\alpha}\overline{\partial}_{\mathsf{i}}$ and a combination of the equations above, we find that

$$\overline{d}^2\overline{\Phi}_{\mathsf{k}}\epsilon_{\mathsf{kij}} = -2i(\overline{\partial}_{\mathsf{i}}\Phi_{\mathsf{j}} - \overline{\partial}_{\mathsf{j}}\Phi_{\mathsf{i}}) = 0, \tag{4.7}$$

which indicates that the massless field equation of the chiral superfield is related to the cohomology of 1-forms. If we set the higher $(\theta^{\mathsf{i}}_\alpha, \overline{\theta}^{\mathsf{i}}_{\dot\alpha})$ components to zero, we have precisely a triplet of chiral and anti-chiral fields. We also need to determine the higher $(\theta^{\mathsf{i}}_\alpha, \overline{\theta}^{\mathsf{i}}_{\dot\alpha})$ components. This is accomplished by computing $d_{\alpha\,\mathsf{i}}\Phi_{\mathsf{j}}$ and $\overline{d}_{\dot\alpha\,\mathsf{i}}\Phi_{\mathsf{j}}$. Using the equations above and the d algebra, we find that

$$d_{\alpha\,\mathsf{i}}\Phi_{\mathsf{j}} = -2i\overline{\partial}_{\mathsf{j}}A_{\alpha\,\mathsf{i}}, \qquad \overline{d}_{\dot\alpha\,\mathsf{i}}\Phi_{\mathsf{j}} = -2i\overline{\partial}_{\mathsf{j}}\overline{A}_{\dot\alpha\,\mathsf{i}}, \tag{4.8}$$

so the higher $(\theta^{\mathsf{i}}_\alpha, \overline{\theta}^{\mathsf{i}}_{\dot\alpha})$ components do not describe new degrees of freedom. If the fields do not depend on $(z^{\mathsf{i}}, \overline{z}^{\mathsf{i}})$ we have a triplet of four-dimensional chiral fields, as desired.

It is easy to check that if we try to impose $A_{\alpha\,\mathsf{i}} = \overline{A}_{\dot\alpha\,\mathsf{i}} = 0$, we get a trivial system. Similarly, a solution where $\Phi_{\mathsf{i}} = 0$ and $\overline{\Phi}_{\mathsf{i}} = 0$ is trivial because the vector field strength W_α is a higher component in Φ. There is no covariant way to solve the constraints containing only the gauge part. However, if the fields do not depend on the internal coordinates, it is possible to isolate the four-dimensional gauge part. Instead of following this path, it is worthwhile to derive the equations of motion from (4.2) for a general $(A_{\alpha\dot\alpha} = i[d_\alpha, \overline{d}_{\dot\alpha}]V, \Phi_{\mathsf{i}}, \overline{\Phi}_{\mathsf{i}})$. Repeated application of the d-algebra gives

$$\overline{d}_{\dot\alpha}(\Phi_{\mathsf{i}} + 2\overline{\partial}_{\mathsf{i}}V) = 0, \qquad d_\alpha(\overline{\Phi}_{\mathsf{i}} - 2\partial_{\mathsf{i}}V) = 0, \tag{4.9}$$

$$d^2\Phi_{\mathsf{i}} + 2i\epsilon_{\mathsf{ijk}}\partial_{\mathsf{j}}\overline{\Phi}_{\mathsf{k}} = 2\overline{\partial}_{\mathsf{i}}d^\alpha A_\alpha, \qquad \overline{d}^2\overline{\Phi}_{\mathsf{i}} + 2i\epsilon_{\mathsf{ijk}}\overline{\partial}_{\mathsf{j}}\Phi_{\mathsf{k}} = 2\partial_{\mathsf{i}}\overline{d}^{\dot\alpha}\overline{A}_{\dot\alpha}, \tag{4.10}$$

$$d^\alpha\overline{d}^2 d_\alpha V - 2\delta^{\mathsf{i}\overline{\mathsf{j}}}(\partial_{\mathsf{i}}\Phi_{\mathsf{j}} - \overline{\partial}_{\mathsf{j}}\overline{\Phi}_{\mathsf{i}}) = 0, \tag{4.11}$$

where higher components of $(\theta^{\mathsf{i}}, \overline{\theta}^{\mathsf{j}})$ (which are consequences of the equations above) are set to zero. These are the linearized equations of motion for ten dimensional superYM in 1+3 notation obtained long ago in reference [17]. If the fields do not depend on the internal coordinates, we get three chiral fields and a vector multiplet. The higher components are again determined by equation (4.2).

4.2 Heterotic string spectrum

The spectrum of the heterotic string is calculated in a similar way by repeated application of the curved space derivative algebra and the equations of motions coming from $QA = 0$. Additionally, we now have to remember that the covariant derivatives act appropriately on each section of the various vector bundles over the CY. We will see that when the section of vertex operator is not in the cohomology of ∇_{i}, the state corresponds to a Kaluza-Klein mode obeying a massive superspace equation of motion.

We begin with the compactification dependent sector. The complete heterotic string vertex operator must be tensored with the right-moving dimension 1 currents[5] given by

[5] In order to get a dimension $(0,0)$ vertex operator we should also multiply by the right-moving ghost $\overline{c}$.

$(\overline{\partial}x^a, \overline{\partial}y^{\mathsf{i}}, \overline{\partial}\overline{y}^{\mathsf{i}}, \overline{\mathbf{J}}_\Sigma)$, where Σ is a general index for the two E_8 algebras. Although not discussed in the present paper,[6] the anomaly cancelation condition of the B-field should be taken into account. The simplest way to solve it is by the standard embedding. This embedding breaks one of the E_8 factors into $E_8 \to E_6 \times \mathrm{SU}(3)$. The Kac-Moody currents are decomposed into

$$\overline{\mathbf{J}}_\Sigma \to (\overline{\mathbf{J}}_\sigma, \overline{\mathbf{J}}_\rho, \overline{\mathbf{J}}_A^{\mathsf{i}}, \overline{\mathbf{J}}_{\overline{A}}^{\overline{\mathsf{j}}}, \overline{\mathbf{J}}^{\mathsf{i}\overline{\mathsf{j}}}) \tag{4.12}$$

where σ is the index of the adjoint representation of E_8, ρ is an index for the adjoint representation of E_6, A is the index for the fundamental representation of E_6, and $(\mathsf{i}\overline{\mathsf{j}})$ are indices for endomorphisms of the holomorphic tangent bundle.

The BRST charge is now

$$Q = \oint (\lambda^\alpha \nabla_\alpha + \overline{\lambda}^{\dot{\alpha}} \overline{\nabla}_{\dot{\alpha}} + \lambda^{\alpha\,\mathsf{i}} \nabla_{\alpha\,\mathsf{i}} + \overline{\lambda}^{\dot{\alpha}\,\mathsf{i}} \overline{\nabla}_{\alpha\,\mathsf{i}}). \tag{4.13}$$

We proceed exactly as in the previous section. The equations from the BRST physical state condition are of the form (4.2) with the operators d replaced by the operators ∇ of (4.13). As in the previous section, we will set $A_\alpha^\Gamma = \overline{A}_{\dot{\alpha}}^\Gamma = 0$ where Γ denotes any right-moving index. After doing this, we obtain the equations

$$\begin{aligned}
\nabla_\alpha A^\Gamma_{\beta\,\mathsf{i}} &= \varepsilon_{\alpha\beta} \overline{\Phi}^\Gamma_{\mathsf{i}}, & \overline{\nabla}_{\dot{\alpha}} A^\Gamma_{\beta\,\mathsf{i}} &= 0 \\
\nabla_{\alpha\,\mathsf{i}} A^\Gamma_{\beta\,\mathsf{j}} + \nabla_{\beta\,\mathsf{j}} A^\Gamma_{\alpha\,\mathsf{i}} &= 2\varepsilon_{\alpha\beta}\epsilon_{\mathsf{ijk}} \Phi^\Gamma_{\mathsf{k}}, & \nabla_{\alpha\,\mathsf{i}} \overline{A}^\Gamma_{\dot{\alpha}\,\mathsf{j}} + \overline{\nabla}_{\dot{\alpha}\,\mathsf{j}} A^\Gamma_{\alpha\,\mathsf{i}} &= 0 \\
\overline{\nabla}_{\dot{\alpha}\,\mathsf{i}} \overline{A}^\Gamma_{\dot{\beta}\,\mathsf{j}} + \overline{\nabla}_{\dot{\beta}\,\mathsf{j}} \overline{A}^\Gamma_{\dot{\alpha}\,\mathsf{i}} &= 2\varepsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\mathsf{ijk}} \overline{\Phi}^\Gamma_{\mathsf{k}} & \overline{\nabla}_{\dot{\alpha}} \overline{A}^\Gamma_{\dot{\beta}\,\mathsf{i}} &= \varepsilon_{\dot{\alpha}\dot{\beta}} \Phi^\Gamma_{\mathsf{i}}
\end{aligned} \tag{4.14}$$

Using these equations and the algebra (3.6) we obtain $\nabla_\alpha \overline{\Phi}^\Gamma_{\mathsf{i}} = 0$ and $\overline{\nabla}^2 \overline{\Phi}^\Gamma_{\mathsf{i}} = 0$.[7] Note that the commutator $[\overline{\nabla}^2, \nabla_{\dot{\alpha}\,\mathsf{i}}] = -4i \nabla_{\dot{\alpha}} \overline{\nabla}_{\mathsf{i}}$ still holds for the covariant derivatives. This implies that the chiral fields Φ^Γ_{i} satisfy

$$\overline{\nabla}_{\mathsf{i}} \Phi^\Gamma_{\mathsf{j}} - \overline{\nabla}_{\mathsf{j}} \Phi^\Gamma_{\mathsf{i}} = 0. \tag{4.15}$$

Thus, for each type of index Γ, the corresponding chiral field is in the cohomology ring $H^{0,1}(\mathfrak{T})$, where $\mathfrak{T}$ is the vector space corresponding to the index Γ. This is the expected result for the matter part of a CY compactification. The analysis of the higher θ-components proceeds as in the previous section. In particular, the Φ^Γ do not describe additional degrees of freedom at the massless level.

To derive the equations of motion for the compactification-independent part, we have to solve the generalization of equations (4.2) with covariant derivatives without setting the superfields A^Γ_α and $\overline{A}^\Gamma_{\dot{\alpha}}$ to zero. Again, we obtain the generalization of (4.9):

$$\overline{\nabla}_{\dot{\alpha}}(\Phi^\Gamma_{\mathsf{i}} + 2\overline{\nabla}_{\mathsf{i}} V^\Gamma) = 0, \qquad \nabla_\alpha(\overline{\Phi}^\Gamma_{\mathsf{i}} - 2\nabla_{\mathsf{i}} V^\Gamma) = 0, \tag{4.16}$$

$$\nabla^2 \Phi^\Gamma_{\mathsf{i}} + 2i\epsilon_{\mathsf{ijk}} \nabla_{\mathsf{j}} \overline{\Phi}^\Gamma_{\mathsf{k}} = 2\overline{\nabla}_{\mathsf{i}} \nabla^\alpha A^\Gamma_\alpha, \qquad \overline{\nabla}^2 \overline{\Phi}^\Gamma_{\mathsf{i}} + 2i\epsilon_{\mathsf{ijk}} \overline{\nabla}_{\mathsf{j}} \Phi^\Gamma_{\mathsf{k}} = 2\nabla_{\mathsf{i}} \nabla^{\dot{\alpha}} A^\Gamma_{\dot{\alpha}}, \tag{4.17}$$

$$\nabla^\alpha \overline{\nabla}^2 \nabla_\alpha V^\Gamma - 2\delta^{\mathsf{i}\overline{\mathsf{j}}}(\nabla_{\mathsf{i}} \Phi^\Gamma_{\mathsf{j}} - \overline{\nabla}_{\mathsf{j}} \overline{\Phi}^\Gamma_{\mathsf{i}}) = 0. \tag{4.18}$$

[6] In the pure spinor formalism this comes from conservation of the BRST current and the anomaly in the conservation of ghost and gauge currents.

[7] We define $\nabla^2 = \nabla^\alpha \nabla_\alpha$ and $\overline{\nabla}^2 = \overline{\nabla}_{\dot{\alpha}} \overline{\nabla}^{\dot{\alpha}}$.

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If the fields do not depend on the compactification, the three possible right moving indices are the four-dimensional vector index, the adjoint E_8 index, and the adjoint E_6 index. This completes the massless spectrum of the heterotic string in the CY background. As a final remark, since the equations above do not impose that the fields are harmonic forms on the CY (see equation 4.17), they also describe in superspace the KK spectrum of the compactification.

5 Discussion

In this paper, we began the study of superstring compactifications using the pure spinor formalism. Although only $N = 1$ supersymmetry in four dimensions is preserved, the description of the BRST operator and spectrum given here uses the full superspace inherited from ten dimensions. We first considered some algebraic aspects of the compactification, mainly the BRST operator and the spectrum. In a second paper we will discuss further aspects, such as the construction of the sigma model describing the dynamics of the compactification and the anomaly cancelation condition, which comes from the conservation of the BRST current. In this discussion the B-field, which played no role in the present work, will be included.

One interesting direction for future work could be to see how the well known non-renormalization theorems of Calabi-Yau compactifications arise in the supersymmetric description given here. This will require knowledge of the zero-mode measure for scattering amplitudes (which will be presented elsewhere). It is possible that the non-renormalization is just a consequence of the superspace integration arising from this measure.

A more important line of research is to generalize these results to Type II strings, especially in the case of flux compactifications (for a review see e.g. [18]). Most of the results in the literature use only supergravity methods and little is known about α' corrections and the spectrum. Even though it is unlikely that a sigma model including all powers of θ can be written explicitly, partial knowledge will already be enough to address important questions pertaining to the form of the effective action of the light modes in a flux compactification. We plan to address flux compactifications of the pure spinor formalism in the future.

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References

[1] N. Berkovits, S. Gukov and B.C. Vallilo, *Superstrings in* 2*D backgrounds with RR flux and new extremal black holes*, *Nucl. Phys.* **B 614** (2001) 195 [hep-th/0107140] [SPIRES].

[2] N. Berkovits, *Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background*, Nucl. Phys. **B 431** (1994) 258 [hep-th/9404162] [SPIRES].

[3] N. Berkovits and C. Vafa, $N = 4$ *topological strings*, Nucl. Phys. **B 433** (1995) 123 [hep-th/9407190] [SPIRES];
N. Berkovits, *Quantization of the type-II superstring in a curved six-dimensional background*, Nucl. Phys. **B 565** (2000) 333 [hep-th/9908041] [SPIRES].

[4] W.D. Linch III and B.C. Vallilo, *Hybrid formalism, supersymmetry reduction and Ramond-Ramond fluxes*, JHEP **01** (2007) 099 [hep-th/0607122] [SPIRES];
W.D. Linch III, J. McOrist and B.C. Vallilo, *Type IIB Flux Vacua from the String Worldsheet*, JHEP **09** (2008) 042 [arXiv:0804.0613] [SPIRES].

[5] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, JHEP **04** (2000) 018 [hep-th/0001035] [SPIRES].

[6] J. Gomis, D. Sorokin and L. Wulff, *The complete* $AdS_4 \times CP^3$ *superspace for the type IIA superstring and D-branes*, JHEP **03** (2009) 015 [arXiv:0811.1566] [SPIRES];
P.A. Grassi, D. Sorokin and L. Wulff, *Simplifying superstring and D-brane actions in* $AdS_4 \times CP^3$ *superbackground*, JHEP **08** (2009) 060 [arXiv:0903.5407] [SPIRES].

[7] D. Tsimpis, *Curved* $11D$ *supergeometry*, JHEP **11** (2004) 087 [hep-th/0407244] [SPIRES].

[8] P.A. Grassi and P. van Nieuwenhuizen, *Harmonic superspaces from superstrings*, Phys. Lett. **B 593** (2004) 271 [hep-th/0402189] [SPIRES].

[9] N. Berkovits, *Pure spinor formalism as an* $N = 2$ *topological string*, JHEP **10** (2005) 089 [hep-th/0509120] [SPIRES].

[10] P.A. Grassi and N. Wyllard, *Lower-dimensional pure-spinor superstrings*, JHEP **12** (2005) 007 [hep-th/0509140] [SPIRES].

[11] O. Chandía, $D = 4$ *pure spinor superstring and* $N = 2$ *strings*, JHEP **10** (2005) 105 [hep-th/0509185] [SPIRES].

[12] I. Adam, P.A. Grassi, L. Mazzucato, Y. Oz and S. Yankielowicz, *Non-critical pure spinor superstrings*, JHEP **03** (2007) 091 [hep-th/0605118] [SPIRES].

[13] N. Berkovits and P.S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, Nucl. Phys. **B 635** (2002) 75 [hep-th/0112160] [SPIRES].

[14] O. Chandía and B.C. Vallilo, *Conformal invariance of the pure spinor superstring in a curved background*, JHEP **04** (2004) 041 [hep-th/0401226] [SPIRES].

[15] O.A. Bedoya and O. Chandía, *One-loop conformal invariance of the type-II pure spinor superstring in a curved background*, JHEP **01** (2007) 042 [hep-th/0609161] [SPIRES].

[16] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library Edition, U.S.A. (1994).

[17] N. Marcus, A. Sagnotti and W. Siegel, *Ten-dimensional supersymmetric Yang-Mills theory in terms of four-dimensional superfields*, Nucl. Phys. **B 224** (1983) 159 [SPIRES].

[18] M. Graña, *Flux compactifications in string theory: a comprehensive review*, Phys. Rept. **423** (2006) 91 [hep-th/0509003] [SPIRES].

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